DILATION PROPERTIES FOR WEIGHTED MODULATION SPACES

ELENA CORDERO AND KASSO A. OKOUDJOU*

Abstract. In this paper we give a sharp estimate on the norm of the scaling operator $U_\lambda f(x) = f(\lambda x)$ acting on the weighted modulation spaces $\mathcal{M}^{p,q}_{s,t}(\mathbb{R}^d)$. In particular, we recover and extend recent results by Sugimoto and Tomita in the unweighted case [14]. As an application of our results, we estimate the growth in time of solutions of the wave and vibrating plate equations, which is of interest when considering the well posedeness of the Cauchy problem for these equations. Finally, we provide new embedding results between modulation and Besov spaces.

1. Introduction

The modulation spaces were introduced by H. Feichtinger [7], by imposing integrability conditions on the short-time Fourier transform (STFT) of tempered distributions. More specifically, for $x, \omega \in \mathbb{R}^d$, we let M_ω and T_x denote the operators of modulation and translation. Then, the STFT of f with respect to a nonzero window g in the Schwartz class is

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^{2d}} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}\,dt.$$

$V_g f(x,\omega)$ measures the frequency content of f in a neighborhood of x.

For $s_1, s_2 \in \mathbb{R}$ and $1 \le p, q \le \infty$, the weighted modulation space $\mathcal{M}^{p,q}_{s_1,s_2}(\mathbb{R}^{2d})$ is defined to be the Banach space of all tempered distributions f such that

$$\|f\|_{\mathcal{M}^{p,q}_{s_1,s_2}} = \left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|V_g f(x,\omega)|^p\, v_{s_1}(x)^p\,dx\right)^{q/p} v_{s_2}(\omega)^q\,d\omega\right)^{1/q} < \infty. \tag{1.1}$$

Here and in the sequel, we use the notation

$$v_s(x) = < x >^s = (1+|x|^2)^{s/2}.$$

The definition of modulation space is independent of the choice of the window g, in the sense that different window functions yield equivalent modulation-space norms. Furthermore, the dual of a modulation space is also a modulation space: if $p < \infty$, $q < \infty$, $(\mathcal{M}^{p,q}_{s,t})' = \mathcal{M}^{p',q'}_{-s,-t}$, where p', q' denote the dual exponents of p and q, respectively.

Date: August 3, 2010.

2000 *Mathematics Subject Classification.* Primary 46E35; Secondary 35S05, 42B35, 47B38, 47G30.

Key words and phrases. Modulation spaces, Besov spaces, Dilation, Inclusion, Wave equations.

*Partially supported by ONR grant: N000140910324, and by a RASA from the Graduate School of UMCP.

When both $s = t = 0$, we will simply write $\mathcal{M}^{p,q} = \mathcal{M}^{p,q}_{0,0}$. The weighted L^2_s space is exactly $\mathcal{M}^{2,2}_{s,0}$, while an application of Plancherel's identity shows that the Sobolev space $\mathcal{H}^s_2$ coincides with $\mathcal{M}^{2,2}_{0,s}$. For further properties and uses of modulation spaces, see Gröchenig's book [9], and we refer to [17] for equivalent definitions of the modulation spaces for all $0 < p, q \leq \infty$.

The modulation spaces appeared in recent years in various areas of mathematics and engineering. Their relationship with other function spaces have been investigated and resulted in embedding results of modulation spaces into other function spaces such as the Besov and Sobolev spaces [10, 14, 15]. Sugimoto and Tomita [14] proved the optimality of certain of the embeddings of modulation spaces into Besov space obtained in [10, 15]. These results were obtained as consequence to optimal bounds of $\|U_\lambda\|_{\mathcal{M}^{p,q}\to\mathcal{M}^{p,q}}$ [14, Theorem 3.1], where $U_\lambda f(\cdot) = f(\lambda\cdot)$ for $\lambda > 0$.

The operator U_λ has been investigated on many other function spaces including the Besov spaces. For purpose of comparison with our results we include the following results summarizing the behavior of U_λ on the Besov spaces [12, Proposition 3]:

Theorem 1.1. *For $\lambda \in (0, \infty)$, $s \in \mathbb{R}$,*

$$C^{-1}\lambda^{-\frac{d}{p}}\min\{1,\lambda^s\}\|f\|_{B^{p,q}_s} \leq \|f_\lambda\|_{B^{p,q}_s} \leq C\lambda^{-\frac{d}{p}}\max\{1,\lambda^s\}\|f\|_{B^{p,q}_s}. \tag{1.2}$$

The estimate on the norm of U_λ on the (unweighted) modulation spaces $\mathcal{M}^{p,q}(\mathbb{R}^d)$ was first obtained by Sugimoto and Tomita [14]. In this paper, we shall derive optimal lower and upper bounds for the operator U_λ on general modulation spaces $\mathcal{M}^{p,q}_{t,s}(\mathbb{R}^d)$. More specifically, the boundedness of U_λ on $\mathcal{M}^{p,q}_{t,s}$ is proved in Theorems 3.1, 3.2 and 3.4, and the optimal bounds on $\|U_\lambda\|_{\mathcal{M}^{p,q}_{t,s}\to\mathcal{M}^{p,q}_{t,s}}$ are established by Theorems 4.12 and 4.13. We wish to point out that it is not trivial to prove sharp bounds on the norm of the operator U_λ, as one has to construct examples of functions in the modulation spaces that achieve the desired optimal estimates. We construct such examples by exploiting the properties of Gabor frames generated by the Gaussian window. It is likely that the functions that we construct can play some role in other areas of analysis where the modulation are used, e.,g., time-frequency analysis of pseudodifferential operators and PDEs.

Interesting applications concern Strichartz estimates for dispersive equations such as the wave equation and the vibrating plate equation on Wiener amalgam and modulation spaces, where the time parameter of the Fourier multiplier symbol is considered as scaling factor. We plan to investigate such applications in a subsequent paper.

Finally, we prove new embeddings between modulation spaces and Besov spaces, generalizing some of the results of [10]. Although strictly speaking this is not an application of the above dilation results, it is clearly in the spirit of the main topic of the present paper, so that we devote a short subsection to the problem.

Our paper is organized as follows. In Section 2 we set up the notation and prove some preliminary results needed to establish our theorems. In Section 3 we prove the complete scaling of weighted modulation spaces. In Section 4 the sharpness of our results are proved, and in Section 5 we point out the applications of our main results.

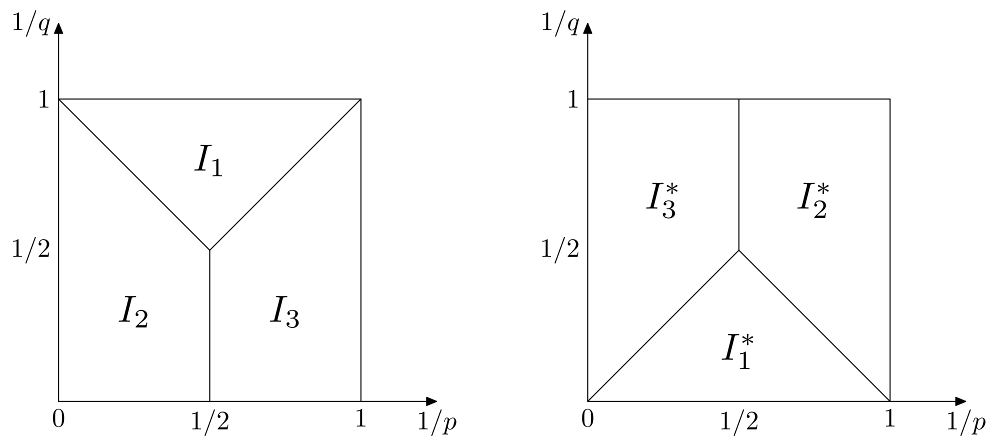
Finally, we shall use the notations $A \lesssim B$ to mean that there exists a constant $c > 0$ such that $A \le cB$, and $A \asymp B$ means that $A \lesssim B \lesssim A$.

2. Preliminary

We shall use the set and index terminology of the paper [14]. Namely, for $1 \le p \le \infty$, let p' be the conjugate exponent of p $(1/p+1/p' = 1)$. For $(1/p, 1/q) \in [0,1]\times[0,1]$, we define the subsets

$$\begin{aligned} I_1 &= \max(1/p, 1/p') \le 1/q, & I_1^* &= \min(1/p, 1/p') \ge 1/q, \\ I_2 &= \max(1/q, 1/2) \le 1/p', & I_2^* &= \min(1/q, 1/2) \ge 1/p', \\ I_3 &= \max(1/q, 1/2) \le 1/p, & I_3^* &= \min(1/q, 1/2) \ge 1/p. \end{aligned}$$

These sets are displayed in Figure 1:



$0 < \lambda \le 1$ $\lambda \ge 1$

Figure 1. The index sets.

We introduce the indices:

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p,1/q) \in I_1^*, \\ 1/q - 1 & \text{if} \quad (1/p,1/q) \in I_2^*, \\ -2/p + 1/q & \text{if} \quad (1/p,1/q) \in I_3^*, \end{cases}$$

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if} \quad (1/p,1/q) \in I_1, \\ 1/q - 1 & \text{if} \quad (1/p,1/q) \in I_2, \\ -2/p + 1/q & \text{if} \quad (1/p,1/q) \in I_3. \end{cases}$$

Next, we prove a lemma that will be used throughout this paper, and which allows us to investigate the action of U_λ only on $\mathcal{S}(\mathbb{R}^d)$.

Lemma 2.1. *Let m be a polynomial growing weight function, A be a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Assume that*

$$\|Af\|_{\mathcal{M}^{p,q}_m} \leq C\|f\|_{\mathcal{M}^{p,q}_m}, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d). \tag{2.1}$$

Then

$$\|Af\|_{\mathcal{M}^{p,q}_m} \leq C\|f\|_{\mathcal{M}^{p,q}_m}, \quad \text{for all } f \in \mathcal{M}^{p,q}_m(\mathbb{R}^d). \tag{2.2}$$

Proof. The conclusion is clear if $p, q < \infty$, because in that case $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{M}^{p,q}_m(\mathbb{R}^d)$.

Consider now the case $p = \infty$ or $q = \infty$. For any given $f \in \mathcal{M}^{p,q}_m$, consider a sequence f_n of Schwartz functions, with $f_n \to f$ in $\mathcal{S}'(\mathbb{R}^d)$, and

$$\|f_n\|_{\mathcal{M}^{p,q}_m} \lesssim \|f\|_{\mathcal{M}^{p,q}_m} \tag{2.3}$$

(see the proof of Proposition 11.3.4 of [9]). Since f_n tends to f in $\mathcal{S}'(\mathbb{R}^d)$, Af_n tends to Af in $\mathcal{S}'(\mathbb{R}^d)$, and $V_\varphi Af_n$ tends to $V_\varphi Af$ pointwise. Hence, by Fatou's Lemma, the assumption (2.1) and (2.3),

$$\|Af\|_{\mathcal{M}^{p,q}_m} \leq \liminf_{n\to\infty} \|Af_n\|_{\mathcal{M}^{p,q}_m} \lesssim \liminf_{n\to\infty} \|f_n\|_{\mathcal{M}^{p,q}_m} \lesssim \|f\|_{\mathcal{M}^{p,q}_m}.$$

□

We shall also make use of the following characterization of the modulation spaces by Gabor frames generated by the Gaussian function, which will be denoted through the paper by $\varphi(x) = e^{-\pi|x|^2}, x \in \mathbb{R}^d$. Recall that for $0 < a < 1$, the family,

$$\mathcal{G}(\varphi, a, 1) = \{\varphi_{k,\ell}(\cdot) = M_\ell T_{ak}\varphi = e^{2\pi i\ell\cdot}\varphi(\cdot - ak), k, \ell \in \mathbb{Z}^d\}$$

is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if there exist $0 < A \leq B < \infty$ such that for all $f \in L^2$ we have

$$A\|f\|^2_{L^2} \leq \sum_{k,\ell\in\mathbb{Z}^d} |\langle f, \varphi_{k,\ell}\rangle|^2 \leq B\|f\|^2_{L^2}. \tag{2.4}$$

Moreover, there exists a dual function $\tilde{\varphi} \in \mathcal{S}$ such that $\mathcal{G}(\tilde{\varphi}, a, 1)$ is also a frame for L^2 and every $f \in L^2$ can be written as

$$f = \sum_{k,\ell\in\mathbb{Z}^d} \langle f, \tilde{\varphi}_{k,\ell}\rangle\varphi_{k,\ell} = \sum_{k,\ell\in\mathbb{Z}^d} \langle f, \varphi_{k,\ell}\rangle\tilde{\varphi}_{k,\ell}. \tag{2.5}$$

It is easy to see from the isometry of the Fourier transform on L^2 and the fact that $\widehat{M_\ell T_{ak}\varphi} = T_\ell M_{-ak}\hat{\varphi} = e^{2\pi iak\ell}M_{-ak}T_\ell\varphi$, that $\mathcal{G}(\varphi, 1, a)$ is a Gabor frame whenever $\mathcal{G}(\varphi, a, 1)$ is. The characterization of the modulation spaces by Gabor frame is summarized in the following proposition. We refer to [9, Chapter 9] for a detail treatment of Gabor frames in the context of the modulation spaces. In particular, the next result is proved in [9, Theorem 7.5.3] and describe precisely when the Gaussian function generates a Gabor frame on L^2.

Proposition 2.2. *$\mathcal{G}(\varphi, a, 1)$ is a Gabor frame for L^2 if and only if $0 < a < 1$. In this case, $\mathcal{G}(\varphi, a, 1)$ is also a Banach frame for $\mathcal{M}^{p,q}_{t,s}$ for all $1 \leq p, q \leq \infty$, and $s, t \in \mathbb{R}$. Moreover, $f \in \mathcal{M}^{p,q}_{t,s}$ if and only if there exists a sequence $\{c_{k,\ell}\}_{k,\ell\in\mathbb{Z}^d} \in \ell^{p,q}_{t,s}(\mathbb{Z}^d \times \mathbb{Z}^d)$ such that $f = \sum_{k,\ell\in\mathbb{Z}^d} c_{k,\ell}\varphi_{k,\ell}$ with convergence in the modulation space norm. In addition,*

$$\|f\|_{\mathcal{M}^{p,q}_{t,s}} \asymp \|c\|_{\ell^{p,q}_{t,s}} := \left(\left(\sum_{k\in\mathbb{Z}^d} |c_{k,\ell}|^p v_t(k)^p\right)^{q/p} v_s(\ell)^q\right)^{1/q}.$$

3. Dilation properties of weighted modulation spaces

We first consider the polynomial weights in the time variables $v_t(x) = \langle x\rangle^t = (1 + |x|^2)^{t/2}$, $t \in \mathbb{R}$.

Theorem 3.1. *Let $1 \leq p, q \leq \infty$, $t \in \mathbb{R}$. Then the following are true:*
(1) There exists a constant $C > 0$ such that $\forall f \in \mathcal{M}^{p,q}_{t,0}$, $\lambda \geq 1$,

$$C^{-1}\, \lambda^{d\mu_2(p,q)} \min\{1, \lambda^{-t}\}\, \|f\|_{\mathcal{M}^{p,q}_{t,0}} \leq \|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{d\mu_1(p,q)} \max\{1, \lambda^{-t}\}\, \|f\|_{\mathcal{M}^{p,q}_{t,0}}. \tag{3.1}$$

(2) There exists a constant $C > 0$ such that $\forall f \in \mathcal{M}^{p,q}_{t,0}$, $0 < \lambda \leq 1$,

$$C^{-1}\, \lambda^{d\mu_1(p,q)} \min\{1, \lambda^{-t}\}\, \|f\|_{\mathcal{M}^{p,q}_{t,0}} \leq \|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{d\mu_2(p,q)} \max\{1, \lambda^{-t}\}\, \|f\|_{\mathcal{M}^{p,q}_{t,0}}. \tag{3.2}$$

Proof. We shall only prove the upper halves of each of the estimates (3.1) and (3.2). The lower halves will follow from the fact that $0 < \lambda \leq 1$ if and only if $1/\lambda \geq 1$ and $f = U_\lambda U_{1/\lambda} f = U_{1/\lambda} U_\lambda f$.

We first consider the case $\lambda \geq 1$. Recall the definition of the dilation operator U_λ given by $U_\lambda f(x) = f(\lambda x)$. Since the mapping $f \mapsto \langle\cdot\rangle^t f$ is an homeomorphism from $\mathcal{M}^{p,q}_{t_0,s}$ to $\mathcal{M}^{p,q}_{t_0-t,s}$, $t, t_0, s \in \mathbb{R}$, see, e.g., [16, Corollary 2.3], we have:

$$\|U_\lambda f\|_{\mathcal{M}^{p,q}_{t,0}} \asymp \|\langle\cdot\rangle^t U_\lambda f\|_{\mathcal{M}^{p,q}}.$$

Using $\langle\cdot\rangle^t U_\lambda f = U_\lambda(\langle\lambda^{-1}\cdot\rangle^t) f$ and the dilation properties for unweighted modulation spaces in [14, Theorem 3.1], we obtain

$$\|U_\lambda(\langle\lambda^{-1}\cdot\rangle^t f)\|_{\mathcal{M}^{p,q}} \leq C\lambda^{d\mu_1(p,q)}\|\langle\lambda^{-1}\cdot\rangle^t f\|_{\mathcal{M}^{p,q}} \asymp \lambda^{d\mu_1(p,q)}\|\langle\cdot\rangle^{-t}\langle\lambda^{-1}\cdot\rangle^t(\langle\cdot\rangle^t f)\|_{\mathcal{M}^{p,q}}.$$

Hence, it remains to prove that the pseudodifferential operator with symbol $g^{(t,\lambda)}(x) := \langle x\rangle^{-t}\langle\lambda^{-1}x\rangle^t$ is bounded on $\mathcal{M}^{p,q}$, and that its norm is bounded above by $\max\{1, \lambda^{-t}\}$. By [9, Theorem 14.5.2], this will follow once we prove that $\|g^{(t,\lambda)}(x)\|_{\mathcal{M}^{\infty,1}} \lesssim \max\{1, \lambda^{-t}\}$. To see this, observe first that

$$|g^{(t,\lambda)}(x)| \lesssim \max\{1, \lambda^{-t}\}, \quad \forall x \in \mathbb{R}^d. \tag{3.3}$$

Indeed, let $v^{(t,\lambda)}(x) = \langle\lambda^{-1}x\rangle^t$. Consider the case $t \geq 0$. Since $\lambda \geq 1$, we have $\lambda^{-1}|x| \leq |x|$ and $v^{(t,\lambda)}(x) \leq \langle x\rangle^t$.

Analogously, for $t < 0$, we have $v^{(t,\lambda)}(x) \leq \lambda^{-t}\langle x\rangle^t$. Consequently, we get the desired estimates (3.3).

Using the inclusion $\mathcal{C}^{d+1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}^{\infty,1}(\mathbb{R}^d)$ we have

$$\|g^{(t,\lambda)}(x)\|_{\mathcal{M}^{\infty,1}} \lesssim \sup_{|\alpha|\leq d+1} \sup_{x\in\mathbb{R}^d} |\partial^\alpha g^{(t,\lambda)}(x)|.$$

By Leibniz' formula, the estimate $|\partial^\beta \langle x\rangle^t| \lesssim \langle x\rangle^{t-|\beta|}$ and (3.3) we see that this last expression is estimated by $\max\{1,\lambda^{-t}\}$.

This concludes the proof of the upper half of (3.1).

We now consider the case $0<\lambda\leq 1$. Observe that by [14] we have

$$\|U_\lambda(\langle\lambda^{-1}\cdot\rangle^t f)\|_{\mathcal{M}^{p,q}} \leq C\lambda^{d\mu_2(p,q)}\|\langle\lambda^{-1}\cdot\rangle^t f\|_{\mathcal{M}^{p,q}} \asymp \lambda^{d\mu_2(p,q)}\|\langle\cdot\rangle^{-t}\langle\lambda^{-1}\cdot\rangle^t(\langle\cdot\rangle^t f)\|_{\mathcal{M}^{p,q}}.$$

Moreover, one easily shows that (3.3) still holds using the same arguments along with the fact that $v^{(t,\lambda)}(x) = \lambda^{-t}(\lambda^2+|x|^2)^{t/2} \leq \lambda^{-t}\langle x\rangle^t$ when $t\geq 0$. Similarly, $v^{(t,\lambda)}(x) \leq \langle x\rangle^t$ when $t<0$. In addition, $g^{(t,\lambda)}(x) = g^{(-t,\lambda^{-1})}(\lambda^{-1}x)$. Hence, by the proof of (3.1) and [14, Theorem 3.1], we see that

$$\|g^{(t,\lambda)}\|_{\mathcal{M}^{\infty,1}} \lesssim \|g^{(-t,\lambda^{-1})}\|_{\mathcal{M}^{\infty,1}} \lesssim \max\{1,\lambda^{-t}\}.$$

This establishes the upper half of (3.2). □

We now consider the polynomial weights in the frequency variables $v_s(\omega) = \langle\omega\rangle^s$, $s\in\mathbb{R}$.

Theorem 3.2. *Let $1\leq p,q\leq\infty$, $s\in\mathbb{R}$. Then the following are true:*
(1) There exists a constant $C>0$ such that $\forall f\in\mathcal{M}^{p,q}_{0,s}$, $\lambda\geq 1$,

$$C^{-1}\,\lambda^{d\mu_2(p,q)}\min\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq \|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \leq C\lambda^{d\mu_1(p,q)}\max\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{0,s}}. \tag{3.4}$$

(2) There exists a constant $C>0$ such that $\forall f\in\mathcal{M}^{p,q}_{0,s}$, $0<\lambda\leq 1$,

$$C^{-1}\,\lambda^{d\mu_1(p,q)}\min\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq \|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \leq C\lambda^{d\mu_2(p,q)}\max\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{0,s}}. \tag{3.5}$$

Proof. Here we use the fact that the mapping $f\mapsto\langle D\rangle^s f$ is an homeomorphism from $\mathcal{M}^{p,q}_{t,s_0}$ to $\mathcal{M}^{p,q}_{t,s_0-s}$, $t,s,s_0\in\mathbb{R}$ (see [16, Corollary 2.3]). The rest of the proof uses similar arguments as those in Theorem 3.1. □

The next result follows immediately by combining the last two theorems.

Corollary 3.3. *Let $1\leq p,q\leq\infty$, $t,s\in\mathbb{R}$. Then the following are true:*
(1) There exists a constant $C>0$ such that $\forall f\in\mathcal{M}^{p,q}_{t,s}$, $\lambda\geq 1$,

$$\begin{aligned} C^{-1}\lambda^{d\mu_2(p,q)}\min\{1,\lambda^{-t}\}\min\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{t,s}} &\leq \|f_\lambda\|_{\mathcal{M}^{p,q}_{t,s}} \\ &\leq C\lambda^{d\mu_1(p,q)}\max\{1,\lambda^{-t}\}\max\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{t,s}}. \end{aligned}$$

(2) There exists a constant $C>0$ such that $\forall f\in\mathcal{M}^{p,q}_{t,s}$, $0<\lambda\leq 1$,

$$\begin{aligned} C^{-1}\lambda^{d\mu_1(p,q)}\min\{1,\lambda^{-t}\}\min\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{t,s}} &\leq \|f_\lambda\|_{\mathcal{M}^{p,q}_{t,s}} \\ &\leq C\lambda^{d\mu_2(p,q)}\max\{1,\lambda^{-t}\}\max\{1,\lambda^s\}\,\|f\|_{\mathcal{M}^{p,q}_{t,s}}. \end{aligned}$$

The following result is an analogue of Corollary 3.3 for modulation spaces defined by non-separable polynomial growing weight function such as $v_s(x,\omega) := \langle(x,\omega)\rangle^s = (1+|x|^2+|\omega|^2)^{s/2}$.

Theorem 3.4. *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$. Then the following are true:*
(1) There exists a constant $C > 0$ such that $\forall f \in \mathcal{M}^{p,q}_{v_s}$, $\lambda \ge 1$,
(3.6)
$$C^{-1}\lambda^{d\mu_2(p,q)} \min\{\lambda^{-s}, \lambda^{s}\} \|f\|_{\mathcal{M}^{p,q}_{v_s}} \le \|f_\lambda\|_{\mathcal{M}^{p,q}_{v_s}} \le C\lambda^{d\mu_1(p,q)} \max\{\lambda^{-s}, \lambda^{s}\} \|f\|_{\mathcal{M}^{p,q}_{v_s}}.$$
(2) There exists a constant $C > 0$ such that $\forall f \in \mathcal{M}^{p,q}_{v_s}$, $0 < \lambda \le 1$,
(3.7)
$$C^{-1}\lambda^{d\mu_1(p,q)} \min\{\lambda^{-s}, \lambda^{s}\} \|f\|_{\mathcal{M}^{p,q}_{v_s}} \le \|f_\lambda\|_{\mathcal{M}^{p,q}_{v_s}} \le C\lambda^{d\mu_2(p,q)} \max\{\lambda^{-s}, \lambda^{s}\} \|f\|_{\mathcal{M}^{p,q}_{v_s}}.$$

Proof. We assume $s \ge 0$. A duality argument can be used to complete the proof when $s < 0$. (Notice, this duality argument will be given explicitly below in the proof of the sharpness of Theorem 3.1 in the case $(1/p, 1/q) \in I_2$, $t \ge 0$).

Moreover, since the result has been proved in [14, Theorem 3.1] for $s = 0$, one can use interpolation arguments along with Lemma 2.1 to reduce the proof when s is an even integer.

The mapping $f \mapsto \langle x, D\rangle^s f$ is an homeomorphism from $\mathcal{M}^{p,q}_{v_s}$ to $\mathcal{M}^{p,q}$, $s \in \mathbb{R}$ (see [16, Theorem 2.2]). Hence

$$\begin{aligned}
\|f_\lambda\|_{\mathcal{M}^{p,q}_{v_s}} &\asymp \|\langle x, D\rangle^s f_\lambda\|_{\mathcal{M}^{p,q}} \\
&= \|U_\lambda(\langle \lambda^{-1}x, \lambda D\rangle^s f)\|_{\mathcal{M}^{p,q}} \\
&\le C \begin{cases} \lambda^{d\mu_1(p,q)} \|\langle \lambda^{-1}x, \lambda D\rangle^s f\|_{\mathcal{M}^{p,q}}, & \lambda \ge 1 \\ \lambda^{d\mu_2(p,q)} \|\langle \lambda^{-1}x, \lambda D\rangle^s f\|_{\mathcal{M}^{p,q}}, & 0 < \lambda \le 1, \end{cases}
\end{aligned}$$

where in the last inequality we used again the dilation properties for unweighted modulation spaces of [14, Theorem 3.1]. Therefore, writing $f = \langle x, D\rangle^{-s}\langle x, D\rangle^s f$ we see that it suffices to prove that the pseudodifferential operator

$$\langle \lambda^{-1}x, \lambda D\rangle^s \langle x, D\rangle^{-s}$$

is bounded on $\mathcal{M}^{p,q}$, and its norm is bounded above by $\max\{1, \lambda^{-s}\}\max\{1, \lambda^{s}\} = \max\{\lambda^{s}, \lambda^{-s}\}$. To this end, we observe that, if s is an even integer, $\langle \lambda^{-1}x, \lambda D\rangle^s$ is a finite sum of operators of the form $\lambda^k x^\alpha D^\beta$, with $|k| \le s$ and $|\alpha| + |\beta| \le s$. Now, Shubin's pseudo-differential calculus [13] shows that the operators $x^\alpha D^\beta \langle x, D\rangle^{-s}$ have bounded symbols, together with all their derivatives, so that they are bounded on $\mathcal{M}^{p,q}$. The proof is completed by taking into account the additional factor λ^k. $\square$

Finally, it is relatively straightforward to give optimal estimates for the dilation operator U_λ on the Wiener amalgam spaces $W(\mathcal{F}L^p_s, L^q_t)$. These spaces are images of modulation spaces under Fourier transform, that is $\mathcal{F}\mathcal{M}^{p,q}_{t,s} = W(\mathcal{F}L^p_s, L^q_t)$. It is also worth noticing that the indices μ_1 and μ_2 obey the following relations,

$$\mu_1(p', q') = -1 - \mu_2(p, q), \quad \mu_2(p', q') = -1 - \mu_1(p, q) \text{ whenever } \tfrac{1}{p} + \tfrac{1}{p'} = \tfrac{1}{q} + \tfrac{1}{q'} = 1.$$

Using the above relations along with the definition of the Wiener amalgam spaces, as well as the behavior of the Fourier transform under dilation, i.e., $\widehat{f_\lambda} = \lambda^{-d}(\hat{f})_{\frac{1}{\lambda}}$ and Corollary 3.3 we obtain the following result

Proposition 3.5. *Let $1 \leq p, q \leq \infty$, $t, s \in \mathbb{R}$. Then the following are true:*
(1) There exists a constant $C > 0$ such that $\forall f \in W(\mathcal{F}L^p_s, L^q_t)$, $\lambda \geq 1$,

$$C^{-1}\lambda^{d\mu_2(p',q')}\min\{1,\lambda^t\}\min\{1,\lambda^{-s}\}\,\|f\|_{W(\mathcal{F}L^p_s,L^q_t)} \leq \|f_\lambda\|_{W(\mathcal{F}L^p_s,L^q_t)} \leq C\lambda^{d\mu_1(p',q')}\max\{1,\lambda^t\}\max\{1,\lambda^{-s}\}\,\|f\|_{W(\mathcal{F}L^p_s,L^q_t)}.$$

(2) There exists a constant $C > 0$ such that $\forall f \in W(\mathcal{F}L^p_s, L^q_t)$, $\lambda \leq 1$,

$$C^{-1}\lambda^{d\mu_1(p',q')}\min\{1,\lambda^t\}\min\{1,\lambda^{-s}\}\,\|f\|_{W(\mathcal{F}L^p_s,L^q_t)} \leq \|f_\lambda\|_{W(\mathcal{F}L^p_s,L^q_t)} \leq C\lambda^{d\mu_2(p',q')}\max\{1,\lambda^t\}\max\{1,\lambda^{-s}\}\,\|f\|_{W(\mathcal{F}L^p_s,L^q_t)}.$$

4. Sharpness of Theorems 3.1 and 3.2.

In this section we prove the sharpness of Theorems 3.1 and 3.2. The sharpness of Theorem 3.4 is proved by modifying the examples constructed in the next subsection. Therefore we omit it. But we first prove some preliminary lemmas in which we construct functions that achieve the optimal bound.

4.1. Preliminary Estimates.

The next two lemmas involve estimates for the modulation space norms of various modifications of the Gaussian. Together with Lemmas 4.3–4.5, they provide examples of functions that achieve the optimal bound under the dilation operator on weighted modulation spaces with weight on the space parameter. Similar constructions for weighted modulation spaces with weight on the frequency parameter are contained in Lemmas 4.6–4.10. Finally, in Lemma 4.11 we investigated the property of the dilation operator on compactly supported functions.

Recall that $\varphi(x) = e^{-\pi|x|^2}$ for $x \in \mathbb{R}^d$, and that $\varphi_\lambda(x) = U_\lambda\varphi(x) = \varphi(\lambda x)$.

Lemma 4.1. *For $t, s \geq 0$, we have*

$$\|\varphi_\lambda\|_{M^{p,q}_{t,0}} \asymp \lambda^{-\frac{d}{p}-t}, \quad 0 < \lambda \leq 1, \tag{4.1}$$

$$\|\varphi_\lambda\|_{M^{p,q}_{t,0}} \asymp \lambda^{-d\left(1-\frac{1}{q}\right)}, \quad \lambda \geq 1, \tag{4.2}$$

$$\|\varphi_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \asymp \lambda^{-\frac{d}{p}}, \quad 0 < \lambda \leq 1, \tag{4.3}$$

and

$$\|\varphi_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \asymp \lambda^{-d\left(1-\frac{1}{q}\right)+s}, \quad \lambda \geq 1. \tag{4.4}$$

Proof. We shall only prove the first two estimates, as the last two are proved similarly. By some straightforward computations, (see, e.g., [9, Lemma 1.5.2]) we get

$$|V_\varphi\varphi_\lambda(x,\omega)| = (\lambda^2+1)^{-\frac{d}{2}}e^{-\pi\frac{\lambda^2}{\lambda^2+1}|x|^2}e^{-\pi\frac{1}{\lambda^2+1}|\omega|^2}. \tag{4.5}$$

Hence

$$\|\varphi_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \asymp \|V_\varphi \varphi_\lambda\|_{\mathcal{M}^{p,q}_{t,0}}$$
$$= p^{-2p} q^{-2q} \lambda^{-\frac{d}{p}} (\lambda^2+1)^{\frac{d}{2}(\frac{1}{p}+\frac{1}{q}-1)} \left(\int_{\mathbb{R}^d} e^{-\pi|x|^2} \langle \frac{\sqrt{\lambda^2+1}}{\lambda\sqrt{p}} x \rangle^{pt}\, dx \right)^{\frac{1}{p}}.$$

If $0<\lambda\le 1$, then

$$\lambda^{-t}|x|^{t/2} \le (\tfrac{\lambda^2+1}{\lambda^2}|x|^2)^{t/2} \le \big(1+\tfrac{\lambda^2+1}{\lambda^2}|x|^2\big)^{t/2} \le 2\lambda^{-t}(1+|x|^2)^{t/2}.$$

Thus, we have

$$\lambda^{-t} \lesssim \left(\int_{\mathbb{R}^d} e^{-\pi|x|^2} \langle \frac{\sqrt{\lambda^2+1}}{\lambda\sqrt{p}} x \rangle^{pt}\, dx \right)^{1/p} \lesssim \lambda^{-t}, \quad 0<\lambda\le 1,$$

and the estimate (4.1) follows.

Now, observe that, if $\lambda \ge 1$, then $\langle \frac{\sqrt{\lambda^2+1}}{\lambda\sqrt{p}} x\rangle \asymp \langle x\rangle$ and (4.2) follows. □

Lemma 4.2. *For $t\le 0$, $\lambda\ge 1$, consider the family of functions*

$$f(x) = \lambda^{-t}\varphi(x-\lambda e_1), \quad e_1=(1,0,0,\dots,0). \tag{4.6}$$

Then there exists a constant $C>0$ such that $\|f\|_{\mathcal{M}^{p,q}_{t,0}} \le C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \gtrsim \lambda^{-t+d(\frac{1}{q}-1)}, \quad \forall\, \lambda \ge 1. \tag{4.7}$$

Proof. We have

$$\|f\|_{\mathcal{M}^{p,q}_{t,0}} \asymp \|V_\varphi f(x,\omega)\langle x\rangle^t\|_{L^{p,q}} = \lambda^{-t}\|V_\varphi\varphi(x-\lambda e_1,\omega)\langle x\rangle^t\|_{L^{p,q}}$$
$$= \lambda^{-t}\|V_\varphi\varphi(x,\omega)\langle x+\lambda e_1\rangle^t\|_{L^{p,q}} \lesssim \lambda^{-t}\lambda^t\|V_\varphi\varphi\langle x\rangle^{-t}\|_{L^{p,q}} \lesssim 1.$$

The last inequality follows from the fact that the weight $\langle\cdot\rangle^t$ is $\langle\cdot\rangle^{-t}$-moderate which implies that $\langle x+\lambda e_1\rangle^t \lesssim \lambda^t\langle x\rangle^{-t}$. This proves the first part of the Lemma. Let us now estimate $\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}}$ from below. We have

$$f_\lambda(x) = \lambda^{-t}\varphi_\lambda(x-e_1).$$

Hence, by arguing as above and using (4.5), we have

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \asymp \lambda^{-t}\|V_\varphi\varphi_\lambda(x,\omega)\langle x+e_1\rangle^t\|_{L^{p,q}}$$
$$\gtrsim \lambda^{-t}\lambda^{d(\frac{1}{q}-1)}\left(\int e^{-\pi p|x|^2}\langle x+e_1\rangle^{pt}dx\right)^{\frac{1}{p}} \gtrsim \lambda^{-t+d(\frac{1}{q}-1)},$$

which concludes the proof. □

Lemma 4.3. *Let $1\le p,q\le\infty$, $\epsilon>0$, $t\in\mathbb{R}$, and $\lambda>1$. Moreover, assume that $(1/p,1/q)\in I_1^*$.*
a) If $t\ge 0$, define

$$f(x) = \sum_{\ell\neq 0} |\ell|^{-d/p-\epsilon} e^{2\pi i\lambda^{-1}\ell\cdot x}\varphi(x) = \sum_{\ell\neq 0}|\ell|^{-d/p-\epsilon} M_{\lambda^{-1}\ell}\varphi(x), \quad \textit{in } \mathcal{S}'(\mathbb{R}^d).$$

Then there exists a constant $C>0$ such that $\|f\|_{\mathcal{M}^{p,q}_{t,0}}\leq C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \gtrsim \lambda^{-d/p-\epsilon}, \qquad \forall\, \lambda>1. \tag{4.8}$$

b) If $t\leq 0$ define

$$f(x)=\sum_{k\neq 0}|k|^{-d/p-\epsilon-t}\varphi(x-k)=\sum_{k\neq 0}|k|^{-d/p-\epsilon-t}T_k\varphi(x), \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

Then there exists a constant $C>0$ such that $\|f\|_{\mathcal{M}^{p,q}_{t,0}}\leq C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \gtrsim \lambda^{-d/p-\epsilon-t}, \qquad \forall\, \lambda>1. \tag{4.9}$$

Proof. We only prove part a) as part b) is obtained similarly. We use Proposition 2.2 to prove that f defined in the lemma belongs to $\mathcal{M}^{p,q}_{t,0}$. Indeed, $\mathcal{G}(\varphi,1,\lambda^{-1})$ is a Gabor frame, and the coefficients of f in this frame are given by $c_{k,\ell}=\delta_{k,0}|\ell|^{-d/p-\epsilon}$ if $\ell\neq 0$ and $c_{0,0}=0$. It is clear that

$$\|c_{k,\ell}\|_{\ell^{p,q}_{t,0}}=\left(\sum_{\ell\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d}|c_{k,\ell}|^p\langle k\rangle^{pt}\right)^{q/p}\right)^{1/q}=\left(\sum_{\ell\neq 0}|\ell|^{q(-d/p-\epsilon)}\right)^{1/q}<\infty,$$

because $q/p\geq 1$. Thus, $f\in\mathcal{M}^{p,q}_{t,0}$ with uniform norm (with respect to λ).

Given $\lambda>1$, we have

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}}=\sup_{\|g\|_{\mathcal{M}^{p',q'}_{-t,0}}=1}|\langle f_\lambda,g\rangle|\geq\|\varphi\|^{-2}_{\mathcal{M}^{p',q'}_{-t,0}}|\langle f_\lambda,\varphi\rangle|.$$

Using relation (4.5),

$$\langle f_\lambda,\varphi\rangle=\sum_{\ell\neq 0}|\ell|^{-d/p-\epsilon}V_\varphi\varphi_\lambda(0,\ell)=\sum_{\ell\neq 0}|\ell|^{-d/p-\epsilon}(1+\lambda^2)^{-d/2}e^{\frac{-\pi|\ell|^2}{\lambda^2+1}}.$$

Therefore, if $\lambda>1$,

$$\begin{aligned}\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}}&\geq C\sum_{\ell\neq 0}|\ell|^{-d/p-\epsilon}(1+\lambda^2)^{-d/2}e^{\frac{-\pi|\ell|^2}{\lambda^2+1}}\\&\geq C\lambda^{-d}\sum_{\ell\neq 0}|\ell|^{-d/p-\epsilon}e^{\frac{-\pi|\ell|^2}{\lambda^2+1}}\\&\geq C\lambda^{-d}\sum_{0<|\ell|<\lambda}|\ell|^{-d/p-\epsilon}e^{\frac{-\pi|\ell|^2}{\lambda^2+1}}\\&\geq C\lambda^{-d}\lambda^{-d/p-\epsilon}\sum_{0<|\ell|<\lambda}e^{-\pi}\\&\geq C\lambda^{-d}\lambda^{-d/p-\epsilon}e^{-\pi}\lambda^d=C\lambda^{-d/p+\epsilon},\end{aligned}$$

from which the proof follows. □

The next results extend [14, Lemma 3.9] and [14, Lemma 3.10].

Lemma 4.4. *Let $1 \le p, q \le \infty$, $t \ge 0$, $\epsilon > 0$. Suppose that $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\operatorname{supp}\psi \subset [-1/2, 1/2]^d$ and $\psi = 1$ on $[-1/4, 1/4]^d$.*
a) If $1 \le q < \infty$, define

$$f(y) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-\frac{d}{q}-\epsilon-t} M_k T_k \psi(y), \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{4.10}$$

Then, $f \in \mathcal{M}^{p,q}_{t,0}(\mathbb{R}^d)$ and

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \gtrsim \lambda^{-d(\frac{2}{p}-\frac{1}{q})+\epsilon-t}, \quad \forall\ 0 < \lambda \le 1. \tag{4.11}$$

b) If $q = \infty$, let

$$f(y) = \sum_{k\neq 0} |k|^{-t} M_k T_k \psi(y), \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{4.12}$$

Then $f \in \mathcal{M}^{p,\infty}_{t,0}$ and

$$\|f_\lambda\|_{\mathcal{M}^{p,\infty}_{t,0}} \gtrsim \lambda^{-\frac{2d}{p}-t}, \quad \forall\ 0 < \lambda \le 1. \tag{4.13}$$

Proof. We only prove part a), i.e., the case $1 \le q < \infty$ as the case $q = \infty$ is proved in a similar fashion.

Let $g \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\operatorname{supp} g \subset [-1/8, 1/8]^d$, and $|\hat{g}| \ge 1$ on $[-2, 2]^d$. The proof of each part of the Lemma is based on the appropriate estimate for $V_g f$.

Let us first show that $f \in \mathcal{M}^{p,q}_{t,0}(\mathbb{R}^d)$. We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} e^{-2\pi i(\omega-k)y}\psi(y-k)g(y-x)dy\right| \\
&= \left|\int_{\mathbb{R}^d} \psi(y-k)g(y-x)\{(1+|\omega-k|^2)^{-d}(I-\Delta_y)^d e^{-2\pi i(\omega-k)y}\}\, dy\right| \\
&= \frac{1}{(1+|\omega-k|^2)^d}\left|\sum_{|\beta_1+\beta_2|\le 2d} C_{\beta_1,\beta_2}\int_{\mathbb{R}^d} \partial^{\beta_1}(T_k\psi)(y)(\partial^{\beta_2}g)(x-y)e^{-2\pi i(\omega-k)y}dy\right| \\
&\le \frac{C}{(1+|\omega-k|^2)^d}\sum_{|\beta_1+\beta_2|\le 2d} (|T_k(\partial^{\beta_1}\psi)| * |\partial^{\beta_2}g|)(x).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|f\|_{\mathcal{M}^{p,q}_{t,0}} \asymp \|V_g f\|_{L^{p,q}_{t,0}} \\
&= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|\sum_{k\neq 0}|k|^{-\frac{d}{q}-\epsilon-t}\int_{\mathbb{R}^d} e^{-2\pi i(\omega-k)y}\psi(y-k)g(y-x)\,dy\right|^p \langle x\rangle^{tp}dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}}
\end{aligned}$$

(4.14)

$$\le C\Big(\int_{\mathbb{R}^d}\big(\sum_{k\neq 0}|k|^{-\frac{d}{q}-\epsilon-t}\frac{1}{(1+|\omega-k|^2)^d}\sum_{|\beta_1+\beta_2|\le 2d}\||T_k(\partial^{\beta_1}\psi)| * |\partial^{\beta_2}g|\|_{L^p_t}\big)^q d\omega\Big)^{\frac{1}{q}}.$$

Using Young's inequality: $\||T_k(\partial^{\beta_1}\psi)| * |\partial^{\beta_2} g|\|_{L^p_t} \lesssim \|T_k\partial^{\beta_1}\psi\|_{L^1_t}\,\|\partial^{\beta_2} g\|_{L^p_t}$, and the estimate $\|T_k\partial^{\beta_1}\psi\|_{L^1_t} \leq \langle k\rangle^t \|\partial^{\beta_1}\psi\|_{L^1_t}$, we can control (4.14) by

$$
\begin{aligned}
& C\left(\int_{\mathbb{R}^d}\left(\sum_{k\neq 0}|k|^{-\frac{d}{q}-\epsilon}\frac{1}{(1+|\omega-k|^2)^d}\right)^q d\omega\right)^{\frac{1}{q}}\\
&\leq C\left(\sum_{\ell\in\mathbb{Z}^d}\int_{\ell+[-1/2,1/2]^d}\left(\sum_{k\neq 0}|k|^{-\frac{d}{q}-\epsilon}\frac{1}{(1+|\omega-k|^2)^d}\right)^q d\omega\right)^{\frac{1}{q}}\\
&\leq \tilde{C}\left(\sum_{\ell\in\mathbb{Z}^d}\left(\sum_{k\neq 0}|k|^{-\frac{d}{q}-\epsilon}\frac{1}{(1+|\ell-k|^2)^d}\right)^q\right)^{\frac{1}{q}}\\
&= \tilde{C}\Big\||k|^{-\frac{d}{q}-\epsilon} * \frac{1}{(1+|k|^2)^d}\Big\|_{\ell^q} < \infty, \qquad (4.15)
\end{aligned}
$$

since $\{|k|^{-\frac{d}{q}-\epsilon}\}_{k\neq 0} \in \ell^q$.

Next, we prove (4.11). Since $V_g f_\lambda(x,\omega) = \lambda^{-d} V_{g_{\lambda^{-1}}} f(\lambda x, \lambda^{-1}\omega)$, we obtain

$$
\|V_g f_\lambda\|_{L^{p,q}_{t,0}} = \lambda^{-d(1+\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_{g_{\lambda^{-1}}} f(x,\omega)|^p\langle\lambda^{-1}x\rangle^{pt}\,dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}}.
$$

Observe that

$$
\langle\lambda^{-1}(x+\ell)\rangle \geq \lambda^{-1}\langle\lambda^{-1}\ell\rangle \geq \langle\lambda^{-1}\ell\rangle
$$

and $\operatorname{supp} g((\cdot-x)/\lambda) \subset \ell+[-1/4,1/4]^d$, for all $0<\lambda\leq 1$, $x\in \ell+[-1/8,1/8]^d$. Since $\operatorname{supp}\psi(\cdot-k)\subset k+[-1/2,1/2]^d$ and $\psi(t-k)=1$ if $t\in k+[-1/4,1/4]^d$, the inner integral can be estimated as follows:

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^d}|V_{g_{\lambda^{-1}}} f(x,\omega)|^p\langle\lambda^{-1}x\rangle^{pt}\,dx\right)^{\frac{1}{p}}\\
&\geq \Big(\sum_{\ell\neq 0}\int_{\ell+[-1/8,1/8]^d}\Big|\sum_{k\neq 0}|k|^{-d/q-\epsilon-t}\int_{\mathbb{R}^d} e^{-2\pi i(\omega-k)y}\psi(y-k)\overline{g(\frac{y-x}{\lambda})}\,dy\Big|^p\langle\lambda^{-1}x\rangle^{pt}\,dx\Big)^{\frac{1}{p}}\\
&\gtrsim \Big(\sum_{\ell\neq 0}(|\ell|^{-\frac{d}{q}-\epsilon-t}\lambda^d|\hat{g}(-\lambda(\omega-\ell))|\,\lambda^{-t}|\ell|^t)^p\Big)^{\frac{1}{p}}\\
&\gtrsim \Big(\sum_{\ell\neq 0}(|\ell|^{-\frac{d}{q}-\epsilon}\lambda^{d-t}|\hat{g}(-\lambda(\omega-\ell))|)^p\Big)^{\frac{1}{p}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|V_g f_\lambda\|_{L^{p,q}_{t,0}} &= \lambda^{-d(1+\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_{g_{\lambda^{-1}}}f(x,\omega)|^p\langle\lambda^{-1}x\rangle^{pt}\,dx\right)^{\frac{q}{p}}d\omega\right)^{\frac{1}{q}}\\
&\gtrsim \lambda^{-d(1+\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{R}^d}\Big(\sum_{\ell\neq 0}(|\ell|^{-\frac{d}{q}-\epsilon}\lambda^{d-t}|\hat{g}(-\lambda(\omega-\ell))|)^p\Big)^{\frac{q}{p}}d\omega\right)^{\frac{1}{q}}\\
&= \lambda^{d-t-d/q}\,\lambda^{-d(1+\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{R}^d}\Big(\sum_{\ell\neq 0}(|\ell|^{-\frac{d}{q}-\epsilon}|\hat{g}(\omega+\lambda\ell)|)^p\Big)^{\frac{q}{p}}d\omega\right)^{\frac{1}{q}}\\
&\gtrsim \lambda^{-t-\frac{d}{p}}\left(\int_{|\omega|\le 1}\Big(\sum_{|\ell|\le\frac{1}{\lambda}}(|\ell|^{-\frac{d}{q}-\epsilon}|\hat{g}(\omega+\lambda\ell)|)^p\Big)^{\frac{q}{p}}d\omega\right)^{\frac{1}{q}}\\
&\gtrsim \lambda^{-t-\frac{d}{p}}\left(\int_{|\omega|\le 1}\Big(\sum_{|\ell|\le\frac{1}{\lambda}}(|\ell|^{-\frac{d}{q}-\epsilon})^p\Big)^{\frac{q}{p}}d\omega\right)^{\frac{1}{q}}\\
&= \lambda^{-t-\frac{d}{p}}\Big(\sum_{|\ell|\le\frac{1}{\lambda}}(|\ell|^{-\frac{d}{q}-\epsilon})^p\Big)^{\frac{1}{p}} \gtrsim \lambda^{-t-\frac{d}{p}}\,\lambda^{\frac{d}{q}+\epsilon}\Big(\sum_{|\ell|\le\frac{1}{\lambda}}\Big)^{\frac{1}{p}} \gtrsim \lambda^{-t-2\frac{d}{p}+\frac{d}{q}+\epsilon},
\end{aligned}
$$

which completes the proof. □

Lemma 4.5. *Let $1 \le p, q \le \infty$ be such that $(1/p, 1/q) \in I_3$. Let $\epsilon > 0$, $t < 0$, and $0 < \lambda < 1$.*
a) If $t \le -d$ define

$$
f(x) = \lambda^{\frac{d}{q}-\frac{2d}{p}+2d}\sum_{k\neq 0}|k|^{-\frac{\epsilon}{2}}T_{\lambda^2 k}\varphi(x), \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{4.16}
$$

Then there exists a constant $C > 0$ such that $\|f\|_{\mathcal{M}^{p,q}_{t,0}} \le C$, uniformly with respect to λ. Moreover,

$$
\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \gtrsim \lambda^{d\mu_2(p,q)+\epsilon}, \quad \forall\, 0 < \lambda < 1. \tag{4.17}
$$

b) If $-d < t < 0$, choose a positive integer N large enough such that $\frac{1}{N} < \frac{p-1}{2} - \frac{pt}{2d}$. Define

$$
f(x) = \lambda^{\frac{d}{q}}\sum_{k\neq 0}|k|^{d(\frac{2}{Np}-1)-\frac{\epsilon}{N}}T_{\lambda^N k}\varphi(x), \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{4.18}
$$

Then the conclusions of part a) still hold.

Proof. a) For the range of p, q being considered, $\frac{d}{q} + 2d - \frac{2d}{p} = d\mu_2(p,q) + 2d \ge 0$, and so if $\lambda < 1$, then $\lambda^{\frac{d}{q}+2d-\frac{2d}{p}} < 1$.

Next, notice that $\mathcal{G}(\varphi, \lambda^2, 1)$ is a Gabor frame. So, to check that $f \in \mathcal{M}^{p,q}_{t,0}$ we only need to verify that the sequence $c = \{c_{k\ell}\} = \{|k|^{-\frac{\epsilon}{2}}\delta_{\ell,0}, k \neq 0\}_{k,\ell\in\mathbb{Z}^d} \in \ell^{p,q}_{t,0}$. But, the condition $t \leq -d$ guarantees this, since

$$\|c\|_{\ell^{p,q}_{t,0}} = \lambda^{\frac{d}{q}+2d-\frac{2d}{p}} \left(\sum_{k\neq 0} |k|^{-p\epsilon/2}(1+|k|^2)^{pt/2} \right)^{1/p} \leq C.$$

Next, as in the proof of Lemma 4.3, we have

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} = \sup_{\|g\|_{\mathcal{M}^{p',q'}_{-t,0}}=1} |\langle f_\lambda, g\rangle| \geq \|\varphi\|^{-2}_{\mathcal{M}^{p',q'}_{-t,0}} |\langle f_\lambda, \varphi\rangle|.$$

In this case,

$$\langle f_\lambda, \varphi\rangle = \lambda^{2d+d\mu_2(p,q)} \sum_{k\neq 0} |k|^{-\epsilon/2} V_\varphi \varphi_\lambda(\lambda k, 0) = \lambda^{2d+d\mu_2(p,q)} \sum_{k\neq 0} |k|^{-\epsilon/2}(1+\lambda^2)^{-d/2} e^{\frac{-\pi\lambda^2|k|^2}{\lambda^2+1}}.$$

Therefore, if $\lambda < 1$,

$$\begin{aligned}
\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} &\geq C\lambda^{2d+d\mu_2(p,q)} \sum_{k\neq 0} |k|^{-\epsilon/2}(1+\lambda^2)^{-d/2} e^{\frac{-\pi\lambda^2|k|^2}{\lambda^2+1}} \\
&\geq C\lambda^{2d+d\mu_2(p,q)} \sum_{k\neq 0} |k|^{-\epsilon/2} \geq C\lambda^{2d+d\mu_2(p,q)} \sum_{0<|k|<\frac{1}{\lambda^2}} |k|^{-\epsilon/2} \\
&\geq C\lambda^{2d+d\mu_2(p,q)} \lambda^{\epsilon}\lambda^{-2d} = C\lambda^{d\mu_2(p,q)+\epsilon}
\end{aligned}$$

which completes the proof of part a).
b) If $p \geq 1$, the assumptions $-d < t < 0$ and $\frac{1}{N} < \frac{p-1}{2} - \frac{pt}{2d}$ are sufficient to prove that $f \in \mathcal{M}^{p,q}_{t,0}$. In addition, the main estimate is that

$$\begin{aligned}
\|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} &\geq C\lambda^{d/q} \sum_{k\neq 0} |k|^{d(\frac{2}{Np}-1)-\frac{\epsilon}{N}} (1+\lambda^2)^{-d/2} e^{\frac{-\pi\lambda^{2(N-1)}|k|^2}{\lambda^2+1}} \\
&\geq C\lambda^{d/q} \sum_{0<|k|<\frac{1}{\lambda^N}} |k|^{d(\frac{2}{Np}-1)-\frac{\epsilon}{N}} \geq C\lambda^{d\mu_2(p,q)+\epsilon}.
\end{aligned}$$

□

We now state results similar to the above lemmas when the weight is in the frequency variable.

Lemma 4.6. *For $s \leq 0$, $0 < \lambda \leq 1$, consider the family of functions*

$$f(x) = \lambda^s M_{\lambda^{-1}e_1}\varphi(x), \;\; e_1 = (1, 0, 0, \ldots, 0). \tag{4.19}$$

Then there exists a constant $C > 0$ such that $\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \gtrsim \lambda^{s-\frac{d}{p}}, \quad \forall\; 0 < \lambda \leq 1. \tag{4.20}$$

Proof. We have

$$\begin{aligned}\|f\|_{\mathcal{M}^{p,q}_{0,s}} &\asymp \|V_\varphi f(x,\omega)\langle\omega\rangle^s\|_{L^{p,q}} = \lambda^s\|V_\varphi\varphi(x,\omega-\lambda^{-1}e_1)\langle\omega\rangle^s\|_{L^{p,q}}\\ &= \lambda^s\|V_\varphi\varphi(x,\omega)\langle\omega+\lambda^{-1}e_1\rangle^s\|_{L^{p,q}} \lesssim \lambda^s\lambda^{-s}\|V_\varphi\varphi\langle\omega\rangle^{-s}\|_{L^{p,q}} \lesssim 1,\end{aligned}$$

where we have used again the fact that the weight $\langle\cdot\rangle^s$ is $\langle\cdot\rangle^{-s}$-moderate. Thus the functions f have norms in $\mathcal{M}^{p,q}_{0,s}$ uniformly bounded with respect to λ. Let us now estimate $\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}}$ from below. We have

$$f_\lambda(x) = \lambda^s M_{e_1}\varphi_\lambda(x).$$

By using (4.5), we obtain

$$\begin{aligned}\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} &= \lambda^s\|V_\varphi\varphi_\lambda(x,\omega-e_1)\langle\omega\rangle^s\|_{L^{p,q}}\\ &\gtrsim \lambda^{s-\frac{d}{p}}\left(\int e^{-\pi q|\omega|^2}\langle\omega+e_1\rangle^{qs}d\omega\right)^{\frac{1}{q}} \gtrsim \lambda^{s-\frac{d}{p}},\end{aligned}$$

as desired. □

Lemma 4.7. *Let $1\le p,q\le\infty$ be such that $(1/p,1/q)\in I_2^*$. Assume that $s\le 0$, $\epsilon>0$ and $\lambda>1$.*
a) If $q\ge 2$ and $s\le 0$, or $1\le q\le 2$ and $s\le -d$, define

$$f(x) = \sum_{\ell\neq 0}|\ell|^{d(\frac{1}{q}-1)-\epsilon}e^{2\pi i\lambda^{-1}\ell\cdot x}\varphi(x) = \sum_{\ell\neq 0}|\ell|^{d(\frac{1}{q}-1)-\epsilon}M_{\lambda^{-1}\ell}\varphi(x), \quad \textit{in } \mathcal{S}'(\mathbb{R}^d).$$

Then there exists a constant $C>0$ such that $\|f\|_{\mathcal{M}^{p,q}_{0,s}}\le C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \gtrsim \lambda^{d(\frac{1}{q}-1)-\epsilon}, \quad \forall\lambda>1. \tag{4.21}$$

b) If $1\le q\le 2$ and $-d<s<0$, choose a positive integer N such that $\frac{1}{N}<-\frac{sq}{d}$, and define

$$f(x) = \sum_{\ell\neq 0}|\ell|^{d(\frac{1}{Nq}-1)-\epsilon/N}e^{2\pi i\lambda^{-N}\ell\cdot x}\varphi(x) = \sum_{\ell\neq 0}|\ell|^{d(\frac{1}{Nq}-1)-\epsilon/N}M_{\lambda^{-N}\ell}\varphi(x), \quad \textit{in } \mathcal{S}'(\mathbb{R}^d).$$

Then the conclusions of part a) still hold.

Proof. a) First of all notice that $\mathcal{G}(\varphi,1,\lambda^{-1})$ is a frame. In addition, $q\ge 2$ is equivalent to $1/q-1\le -1/q$. Thus, for all $s\le 0$, $\{|\ell|^{d(1/q-1)-\epsilon},\ell\neq 0\}\in\ell^q_s$, which ensures that the function f defined above belongs to $\mathcal{M}^{p,q}_{0,s}$. This is also true when $1\le q\le 2$ and $s\le -d$.

To prove (4.21) we follow the proof of Lemma 4.3. In particular, we have

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \ge C\sum_{\ell\neq 0}|\ell|^{d(\frac{1}{q}-1)-\epsilon}(1+\lambda^2)^{-d/2}e^{\frac{-\pi|\ell|^2}{\lambda^2+1}} \ge C\lambda^{-d}\sum_{0<|\ell|\le\lambda}|\ell|^{d(1/q-1)-\epsilon}e^{\frac{-\pi|\ell|^2}{\lambda^2+1}},$$

from which (4.21) follows.

b) In this case, $\mathcal{G}(\varphi, 1, \lambda^{-N})$ is a frame. Moreover, the choice of N insures that $d(1/(Nq)-1)+s<-d$ which is enough to prove that $f \in \mathcal{M}^{p,q}_{0,s}$, and that $\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C$. Relation (4.21) now follows from

$$\begin{aligned}\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} &\geq C \sum_{\ell \neq 0} |\ell|^{d(\frac{1}{Nq}-1)-\epsilon/N} (1+\lambda^2)^{-d/2} e^{\frac{-\pi\lambda^{-2N+2}|\ell|^2}{\lambda^2+1}} \\ &\geq C\lambda^{-d} \sum_{0<|\ell|\leq \lambda^N} |\ell|^{d(\frac{1}{Nq}-1)-\epsilon/N} e^{\frac{-\pi\lambda^{-2N+2}|\ell|^2}{\lambda^2+1}} \geq C\lambda^{d(\frac{1}{q}-1)-\epsilon}.\end{aligned}$$

□

The next lemma is proved similarly to Lemma 4.4, so we omit its proof.

Lemma 4.8. *Let $1 \leq p, q \leq \infty$, $s \leq 0$, $\epsilon > 0$. Suppose that $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfies* $\operatorname{supp}\psi \subset [-1/2, 1/2]^d$ *and $\psi = 1$ on $[-1/4, 1/4]^d$.*
a) If $1 \leq q < \infty$, define $f(y) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{-\frac{d}{q}-\epsilon-s} M_k T_k \psi(y)$, in $\mathcal{S}'(\mathbb{R}^d)$. Then, $f \in \mathcal{M}^{p,q}_{0,s}(\mathbb{R}^d)$ and

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \gtrsim \lambda^{-d(\frac{2}{p}-\frac{1}{q})+\epsilon+s}, \quad \forall\, 0<\lambda \leq 1. \tag{4.22}$$

b) If $q = \infty$, let

$$f(y) = \sum_{k \neq 0} |k|^{-s} e^{2\pi i k y} T_k \psi(y), \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{4.23}$$

Then $f \in \mathcal{M}^{p,\infty}_{0,s}$ and

$$\|f_\lambda\|_{\mathcal{M}^{p,\infty}_{0,s}} \gtrsim \lambda^{-\frac{2d}{p}+s}, \quad \forall\, 0<\lambda \leq 1. \tag{4.24}$$

Lemma 4.9. *Let $1 \leq p, q \leq \infty$ be such that $(1/p, 1/q) \in I_3$. Let $\epsilon > 0$, $s \geq 0$ and $0 < \lambda < 1$. Assume that $p > 1$, and choose a positive integer N such that $\frac{1}{N} < \frac{p-1}{2}$. Define*

$$f(x) = \lambda^{\frac{d}{q}} \sum_{k \neq 0} |k|^{d(\frac{2}{Np}-1)-\frac{\epsilon}{N}} T_{\lambda^N k} \varphi(x), \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{4.25}$$

Then, there exists a constant $C > 0$ such that $\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \gtrsim \lambda^{d\mu_2(p,q)+\epsilon}.$$

Proof. In this case, $\mathcal{G}(\varphi, \lambda^N, 1)$ is a frame. The condition $\frac{1}{N} < \frac{p-1}{2}$ is equivalent to $\frac{2}{Np} - 1 < -\frac{1}{p}$ which is enough to show that $\{|k|^{d(\frac{2}{Np}-1)-\frac{\epsilon}{N}}\}_{k \neq 0} \in \ell^p$. Therefore, $f \in \mathcal{M}^{p,q}_{0,s}$ with $\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C$ where C is a universal constant. The rest of the proof is an adaptation of the proof of Lemma 4.5. □

Notice that, the previous lemma excludes the case $p = 1$. We prove this last case by considering the dual case. Observe that the case $(1/\infty, 1/\infty) \in I_1^* \cap I_3^*$ was already considered in dealing with the region I_1^*.

Lemma 4.10. *Let $1 \leq q \leq \infty$ be such that $(1/\infty, 1/q) \in I_3^*$. Let $\epsilon > 0$, $s \leq 0$ and $\lambda > 1$.*
a) If $1 < q < 2$, choose a positive integer N such that $\frac{3}{N} < q - 1$. Define

$$f(x) = \lambda^{d(1-\frac{2}{q})} \sum_{\ell \neq 0} |\ell|^{d(\frac{3}{Nq}-1)-\frac{\epsilon}{N}} M_{\lambda^{-N}\ell}\varphi(x), \quad in\ \mathcal{S}'(\mathbb{R}^d). \tag{4.26}$$

Then there exists a constant $C > 0$ such that $\|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{\infty,q}_{0,s}} \gtrsim \lambda^{\frac{d}{q}-\epsilon}.$$

b) If $2 \leq q < \infty$, choose a positive integer N such that $N > 2 + q$. Define

$$f(x) = \lambda^{d+\frac{d(2-N)}{q}} \sum_{\ell \neq 0} |\ell|^{d(\frac{N-1}{Nq}-1)-\frac{\epsilon}{N}} M_{\lambda^{-N}\ell}\varphi(x), \quad in\ \mathcal{S}'(\mathbb{R}^d). \tag{4.27}$$

Then the conclusions of part a) still hold.
c) If $q = 1$ and $s \leq -d$, define

$$f(x) = \sum_{\ell \neq 0} |\ell|^{-\frac{\epsilon}{2}} M_{\lambda^{-2}\ell}\varphi(x), \quad in\ \mathcal{S}'(\mathbb{R}^d). \tag{4.28}$$

Then there exists a constant $C > 0$ such that $\|f\|_{\mathcal{M}^{\infty,1}_{0,s}} \leq C$, uniformly with respect to λ. Moreover,

$$\|f_\lambda\|_{\mathcal{M}^{\infty,1}_{0,s}} \gtrsim \lambda^{d-\epsilon}.$$

d) If $q = 1$ and $-d < s < 0$, choose a positive integer N such that $\frac{1}{N} < \frac{-s}{2d}$. Define

$$f(x) = \sum_{\ell \neq 0} |\ell|^{d(\frac{2}{N}-1)-\frac{\epsilon}{N}} M_{\lambda^{-N}\ell}\varphi(x), \quad in\ \mathcal{S}'(\mathbb{R}^d). \tag{4.29}$$

Then the conclusions of part c) still hold.

Proof. a) In this case, $\mathcal{G}(\varphi, 1, \lambda^{-N})$ is a frame. The hypotheses $1 < q < 2$ and $\lambda > 1$ imply that $\lambda^{d(1-\frac{2}{q})} < 1$. In addition, the condition $\frac{3}{N} < q - 1$ is equivalent to $\frac{3}{Nq} - 1 < -\frac{1}{q}$ which is enough to show that $\{|\ell|^{d(\frac{3}{Nq}-1)-\frac{\epsilon}{N}}\}_{\ell \neq 0} \in \ell^q_s$. Therefore, $f \in \mathcal{M}^{\infty,q}_{0,s}$ with $\|f\|_{\mathcal{M}^{\infty,q}_{0,s}} \leq C$ where C is a universal constant. The rest of the proof is an adaptation of the proof of Lemma 4.5.
b) Assume that $2 \leq q < \infty$. The proof is similar to the above with the following differences: $N > q+2$ and $\lambda > 1$ imply that $\lambda^{d(1+\frac{2-N}{q})} < 1$. In addition, the condition $q \geq 2$ implies that $\frac{N-1}{Nq} - 1 < -\frac{1}{q}$. This is enough to show that $\{|\ell|^{d(\frac{N-1}{Nq}-1)-\frac{\epsilon}{N}}\}_{\ell \neq 0} \in \ell^q_s$. Therefore, $f \in \mathcal{M}^{\infty,q}_{0,s}$ with $\|f\|_{\mathcal{M}^{\infty,q}_{0,s}} \leq C$ where C is a universal constant.
c) In this case, $\mathcal{G}(\varphi, 1, \lambda^{-2})$ is a frame. The fact that $s \leq -d$ implies that $\{|\ell|^{-\frac{\epsilon}{2}}\}_{\ell \neq 0} \in \ell^1_s$. Therefore, $f \in \mathcal{M}^{\infty,1}_{0,s}$ with $\|f\|_{\mathcal{M}^{\infty,1}_{0,s}} \leq C$ where C is a universal constant. The rest of the proof is an adaptation of the proof of Lemma 4.5.

d) In this case, $\mathcal{G}(\varphi, 1, \lambda^{-N})$ is a frame. The fact that $-d < s < 0$ and the choice of N imply that $d(\frac{2}{N} - 1) + s < -d$. Therefore $\{|\ell|^{d(\frac{2}{N}-1)-\frac{\epsilon}{2}}\}_{\ell\neq 0} \in \ell^1_s$. Therefore, $f \in \mathcal{M}^{\infty,1}_{0,s}$ with $\|f\|_{\mathcal{M}^{\infty,1}_{0,s}} \leq C$ where C is a universal constant. The rest of the proof is an adaptation of the proof of Lemma 4.5.

□

We finish this subsection by proving lower bound estimates for the dilation of functions that are compactly supported either in the time or in the frequency variables.

Lemma 4.11. *Let $u \in \mathcal{S}(\mathbb{R}^d)$, $\lambda \in (0, \infty)$ and $1 \leq p, q \leq \infty$.*

(i) If u is supported in a compact set $K \subset \mathbb{R}^d$, then, for every $t \in \mathbb{R}$, and $\lambda \geq 1$

$$\|u_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \gtrsim \lambda^{-d(1-\frac{1}{q})} \min\{1, \lambda^{-t}\}. \tag{4.30}$$

(ii) If $\hat{u}$ is supported in a compact set $K \subset \mathbb{R}^d$, then, for every $s \in \mathbb{R}$, and $\lambda \leq 1$

$$\|u_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \gtrsim C\lambda^{-\frac{d}{p}} \min\{1, \lambda^{s}\}. \tag{4.31}$$

Proof. We use the dilation properties for the Sobolev spaces (Bessel potential spaces) $H^p_s(\mathbb{R}^d)$ (see, e.g., [12, Proposition 3]):

$$C^{-1}\lambda^{-\frac{d}{p}} \min\{1, \lambda^s\}\|u\|_{H^p_s} \leq \|u_\lambda\|_{H^p_s} \leq C\lambda^{-\frac{d}{p}} \max\{1, \lambda^s\}\|u\|_{H^p_s}, \quad 1 \leq p \leq \infty,\ s > 0.$$

(i) Let u be supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in \mathcal{M}^{p,q} \Leftrightarrow u \in \mathcal{F}L^q$, and

$$C_K^{-1}\|u\|_{\mathcal{M}^{p,q}} \leq \|u\|_{\mathcal{F}L^q} \leq C_K\|u\|_{\mathcal{M}^{p,q}}, \tag{4.32}$$

where $C_K > 0$ depends only on K (see, e.g., [8, 11]). Hence, if $\lambda \geq 1$,

$$\begin{aligned}\|u_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} &\asymp \|\langle\cdot\rangle^t u_\lambda\|_{\mathcal{M}^{p,q}} \asymp \|\langle\cdot\rangle^t u_\lambda\|_{\mathcal{F}L^q} \asymp \|\mathcal{F}^{-1}(u_\lambda)\|_{H^q_t} = \lambda^{-d}\|(\mathcal{F}^{-1}u)_{\lambda^{-1}}\|_{H^q_t}\\ &\geq \lambda^{-d}(\lambda^{-1})^{-\frac{d}{q}} \min\{1, \lambda^{-t}\}.\end{aligned}$$

(ii) Now let $\hat{u}$ be supported in a compact set $K \subset \mathbb{R}^d$. We have $u \in \mathcal{M}^{p,q} \Leftrightarrow u \in L^p$, and

$$C_K^{-1}\|u\|_{\mathcal{M}^{p,q}} \leq \|u\|_{L^p} \leq C_K\|u\|_{\mathcal{M}^{p,q}},$$

where $C_K > 0$ depends only on K (again, see, e.g., [8]). Arguing as in part (i) above with $0 < \lambda \leq 1$,

$$\|u_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \asymp \|\langle D\rangle^s u_\lambda\|_{\mathcal{M}^{p,q}} \asymp \|\langle D\rangle^s u_\lambda\|_{L^p} \asymp \|u_\lambda\|_{H^p_s} \geq C\lambda^{-\frac{d}{p}} \min\{1, \lambda^s\}\|u\|_{H^p_s}$$

and the proof is completed. □

4.2. **Sharpness of Theorems 3.1 and 3.2.** We are now in position to state and prove the sharpness of the results obtained in Section 3. In particular, Theorem 3.1 is optimal in the following sense:

Theorem 4.12. *Let $1 \leq p, q \leq \infty$.*
(A) If $t \geq 0$ then the following statements hold:
Assume that there exist constants $C > 0$, and $\alpha, \beta \in \mathbb{R}$ such that

$$C^{-1}\lambda^{\beta}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{\alpha}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \quad \forall f \in \mathcal{M}^{p,q}_{t,0} \quad \textit{and} \quad \lambda \geq 1, \tag{4.33}$$

then, $\alpha \geq d\mu_1(p,q)$*, and* $\beta \leq d\mu_2(p,q) - t$*.*
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^{\alpha}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{\beta}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \quad \forall f \in \mathcal{M}^{p,q}_{t,0} \quad \text{and} \quad 0 < \lambda \leq 1, \tag{4.34}$$

then, $\alpha \geq d\mu_1(p,q)$*, and* $\beta \leq d\mu_2(p,q) - t$*.*
(B) If $t \leq 0$ *then the following statements hold:*
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^{\beta}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{\alpha}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \quad \forall f \in \mathcal{M}^{p,q}_{t,0} \quad \text{and} \quad \lambda \geq 1, \tag{4.35}$$

then, $\alpha \geq d\mu_1(p,q) - t$*, and* $\beta \leq d\mu_2(p,q)$*.*
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^{\alpha}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{\beta}\|f\|_{\mathcal{M}^{p,q}_{t,0}} \quad \forall f \in \mathcal{M}^{p,q}_{t,0} \quad \text{and} \quad 0 < \lambda \leq 1, \tag{4.36}$$

then, $\alpha \geq d\mu_1(p,q) - t$*, and* $\beta \leq d\mu_2(p,q)$*.*

Proof. It will be enough to prove the upper half of each of the estimates, as the lower halves will follow from the fact that $f = U_\lambda U_{1/\lambda} f$. Moreover, the proof relies on analyzing the examples provided by the previous lemmas, and by considering several cases.

Case 1: $(1/p, 1/q) \in I_2^*$**,** $t \geq 0$**.** In this case we have $\lambda \geq 1$ and $\mu_1(p,q) = 1/q - 1$. Substitute $f(x) = \varphi(x) = e^{-\pi|x|^2}$ in the upper half estimates (4.33) and use Lemma 4.1 to obtain

$$\lambda^{-d(1-1/q)} \lesssim \|\varphi_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{\alpha}\|\varphi\|_{\mathcal{M}^{p,q}_{t,0}},$$

for all $\lambda \geq 1$. This immediately implies that $\alpha \geq -d(1 - 1/q) = d\mu_1(p,q)$.

Case 2: $(1/p, 1/q) \in I_2$**,** $t \leq 0$**.** This is the dual case to the previous case and can be handled as follows. In this case we have $\lambda \leq 1$ and $\mu_2(p,q) = 1/q - 1$. Assume that the upper-half estimate in (4.36) holds. Notice that $(1/p, 1/q) \in I_2$ if and only if $(1/p', 1/q') \in I_2^*$, and that $\lambda \leq 1$ if and only if $1/\lambda \geq 1$.

$$\begin{aligned}\|f_{1/\lambda}\|_{\mathcal{M}^{p',q'}_{-t,0}} &= \sup|\langle f_{1/\lambda}, g\rangle| = \lambda^d \sup|\langle f, g_\lambda\rangle| \leq \lambda^d \|f\|_{\mathcal{M}^{p',q'}_{-t,0}} \sup\|g_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \\ &\leq \lambda^{d+\beta}\|f\|_{\mathcal{M}^{p',q'}_{-t,0}} \sup\|g\|_{\mathcal{M}^{p,q}_{t,0}},\end{aligned}$$

where the supremum is taken over all $g \in \mathcal{S}$ and $\|g\|_{\mathcal{M}^{p,q}_{t,0}} = 1$; hence,

$$\|f_{1/\lambda}\|_{\mathcal{M}^{p',q'}_{-t,0}} \leq \lambda^{d+\beta}\|f\|_{\mathcal{M}^{p',q'}_{-t,0}}.$$

Thus from Case 1 above, $-\beta - d \geq d\mu_1(p', q') = d/q' - d$. Hence, $\beta \leq d\mu_2(p,q)$.

Case 3: $(1/p, 1/q) \in I_3$**,** $t \geq 0$**.** In this case we have $\lambda \leq 1$ and $\mu_2(p,q) = -2/p + 1/q$. First assume that $1 \leq q < \infty$ and that the upper-half estimate in (4.35) holds for all $f \in \mathcal{M}^{p,q}_{t,0}$ and $0 < \lambda < 1$, but that $\beta > d\mu_2(p,q) - t$. Then there is $\epsilon > 0$ such that $\beta > d\mu_2(p,q) - t + \epsilon$. For this choice of $\epsilon > 0$, we construct a function f as in (4.10) of Lemma 4.4 such that:

$$\lambda^{d\mu_2(p,q) - t + \epsilon} \lesssim \|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^{\beta}\|f\|_{\mathcal{M}^{p,q}_{t,0}}$$

for some $C > 0$ and all $0 < \lambda \leq 1$. This leads to a contradiction on the choice of ϵ.

When $q = \infty$ the function given by (4.12) of Lemma 4.4 gives the optimal bound.

Case 4**:** $(1/p, 1/q) \in I_3^*$**,** $t \leq 0$**.** In this case, $\lambda \geq 1$, and $\mu_1(p,q) = -2/p + 1/q$. This is the dual of Case 3, and a duality argument similar to the used in Case 2 above gives the result.

Case 5**:** $(1/p, 1/q) \in I_1^*$**,** $t \leq 0$**.** In this case, $\lambda \geq 1$, and $\mu_1(p,q) = -1/p$. Assume that the upper-half estimate in (4.35) holds and that $\alpha < d\mu_1(p,q) - t$. Then, choose $\epsilon > 0$ and construct a function f as in part *b*) of Lemma 4.3. A contradiction immediately follows.

Case 6**:** $(1/p, 1/q) \in I_1$**,** $t \geq 0$**.** In this case $\lambda \leq 1$, and $\mu_2(p,q) = -1/p$. This is the dual of Case 5.

Case 7**:** $(1/p, 1/q) \in I_1^*$**,** $t \geq 0$**.** In this case $\lambda \geq 1$, and $\mu_1(p,q) = -1/p$. Assume that the upper-half estimate in (4.33) holds for all $f \in \mathcal{M}^{p,q}_{t,0}$ and $\lambda > 1$, but that $\alpha < d\mu_1(p,q)$. Then there is $\epsilon > 0$ such that $\alpha < d\mu_1(p,q) - \epsilon$. For this choice of $\epsilon > 0$, we can now construct a function f as in Lemma 4.3, part *a*), such that:

$$\lambda^{d\mu_1(p,q)-\epsilon} \lesssim \|f_\lambda\|_{\mathcal{M}^{p,q}_{t,0}} \leq C\lambda^\alpha \|f\|_{\mathcal{M}^{p,q}_{t,0}}$$

for some $C > 0$ and all $\lambda \geq 1$. This leads to a contradiction on the choice of ϵ.

Case 8**:** $(1/p, 1/q) \in I_1$**,** $t \leq 0$**.** In this case $\lambda \leq 1$, and $\mu_2(p,q) = -1/p$. This is the dual of Case 7.

Case 9**:** $(1/p, 1/q) \in I_2^*$**,** $t \leq 0$**.** In this case $\lambda \geq 1$, and $\mu_1(p,q) = 1/q - 1$. The function constructed in Lemma 4.2 leads to the result.

Case 10**:** $(1/p, 1/q) \in I_2$**,** $t \geq 0$**.** In this case $\lambda \leq 1$, and $\mu_2(p,q) = 1/q - 1$. This is the dual of Case 9.

Case 11**:** $(1/p, 1/q) \in I_3$**,** $t \leq 0$**.** In this case $\lambda \leq 1$, and $\mu_1(p,q) = -2/p + 1/q$ and Lemma 4.5 can be used to conclude.

Case 12**:** $(1/p, 1/q) \in I_3^*$**,** $t \geq 0$**.** In this case $\lambda \geq 1$, and $\mu_2(p,q) = -2/p + 1/q$. This is the dual of Case 11. □

We next consider the sharpness Theorem 3.2.

Theorem 4.13. *Let* $1 \leq p, q \leq \infty$.
(A) If $s \geq 0$ *then the following statements hold:*
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^\beta \|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C\lambda^\alpha \|f\|_{\mathcal{M}^{p,q}_{0,s}} \quad \forall f \in \mathcal{M}^{p,q}_{0,s} \quad \text{and} \quad \lambda \geq 1, \tag{4.37}$$

then, $\alpha \geq d\mu_1(p,q) + s$*, and* $\beta \leq d\mu_2(p,q)$.
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^\alpha \|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C\lambda^\beta \|f\|_{\mathcal{M}^{p,q}_{0,s}} \quad \forall f \in \mathcal{M}^{p,q}_{0,s} \quad \text{and} \quad 0 < \lambda \leq 1, \tag{4.38}$$

then, $\alpha \geq d\mu_1(p,q) + s$*, and* $\beta \leq d\mu_2(p,q)$.
(B) If $s \leq 0$ *then the following statements hold:*
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^\beta \|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C\lambda^\alpha \|f\|_{\mathcal{M}^{p,q}_{0,s}} \quad \forall f \in \mathcal{M}^{p,q}_{0,s} \quad \text{and} \quad \lambda \geq 1, \tag{4.39}$$

then, $\alpha \geq d\mu_1(p,q)$*, and* $\beta \leq d\mu_2(p,q) + s$.
Assume that there exist constants $C > 0$*, and* $\alpha, \beta \in \mathbb{R}$ *such that*

$$C^{-1}\lambda^\alpha \|f\|_{\mathcal{M}^{p,q}_{0,s}} \leq \|U_\lambda f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C\lambda^\beta \|f\|_{\mathcal{M}^{p,q}_{0,s}} \quad \forall f \in \mathcal{M}^{p,q}_{0,s} \quad \text{and} \quad 0 < \lambda \leq 1, \tag{4.40}$$

then, $\alpha \geq d\mu_1(p,q)$*, and* $\beta \leq d\mu_2(p,q) + s$.

Proof. As for the time weights, it is enough to prove the upper half of each estimates. Moreover, in what follows we consider only 6 of the 12 cases to be proved, since the others are obtained by the same duality argument used in the previous theorem.

Case 1**:** $(1/p, 1/q) \in I_1$**,** $s \geq 0$**.** In this case, $0 < \lambda \leq 1$ and $\mu_2(p,q) = -1/p$. Assume there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that the upper-half estimate (4.38) holds. Taking the Gaussian $f = \varphi$ as in Lemma 4.1 and using (4.3), we have

$$\lambda^{-\frac{d}{p}} \lesssim \|\varphi_\lambda\|_{\mathcal{M}^{p,q}_{0,s}} \lesssim \lambda^{\beta} \|\varphi\|_{\mathcal{M}^{p,q}_{0,s}},$$

for all $0 < \lambda \leq 1$. This gives $\beta \leq -d/p$.

Case 2**:** $(1/p, 1/q) \in I_1$**,** $s \leq 0$**.** Here $\lambda \leq 1$ and we test the upper-half estimate (4.40) on the family of functions (4.19). Using (4.20), we obtain $\beta \leq s - d/p$.

Case 3**:** $(1/p, 1/q) \in I_2^*$**,** $s \geq 0$**.** Here $\lambda \geq 1$, $\mu_1(p,q) = 1/q - 1$. We assume the upper-half estimate (4.37) and test it on the dilated Gaussian function in (4.4), obtaining $\alpha \geq d(1/q - 1) + s$.

Case 4**:** $(1/p, 1/q) \in I_2^*$**,** $s \leq 0$**.** Here $\lambda \geq 1$, $\mu_1(p,q) = 1/q - 1$. We use a contradiction argument based on Lemma 4.7.

Case 5**:** $(1/p, 1/q) \in I_3$**,** $s \geq 0$**.** Here $\lambda \leq 1$, $\mu_2(p,q) = -2/p + 1/q$. The sharpness is obtained by testing the upper-half estimate (4.38) on the family of functions f_λ, defined in Lemma 4.9 when $p > 1$.

If $p = 1$ we consider the dual case, that is $(1/\infty, 1/q) \in I_3^*$, $s \leq 0$. Here $\lambda \geq 1$, $\mu_1(\infty, q) = 1/q$. We use a contradiction argument based on Lemma 4.10.

Case 6**:** $(1/p, 1/q) \in I_3$**,** $s \leq 0$**.** Here $\lambda \leq 1$, $\mu_2(p,q) = -2/p + 1/q$. The sharpness is obtained by testing the upper-half estimate (4.40) on the family of functions f_λ, defined in Lemma 4.8.

□

5. Applications

5.1. **Applications to dispersive equations.**

5.1.1. *Wave equation.* Let us first recall the Cauchy problem for the wave equation:

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 \\ u(0,x) = u_0(x), \;\; \partial_t u(0,x) = u_1(x), \end{cases} \tag{5.1}$$

with $t \geq 0$, $x \in \mathbb{R}^d$, $d \geq 1$, $\Delta_x = \partial^2_{x_1} + \dots \partial^2_{x_d}$. The formal solution $u(t,x)$ is given by

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} e^{2\pi i x\xi} \cos(2\pi t|\xi|) \widehat{u_0}(\xi)\, d\xi + \int_{\mathbb{R}^d} e^{2\pi i x\xi} \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|} \widehat{u_1}(\xi)\, d\xi, \\ &= H_{\sigma_0} u_0(x) + H_{\sigma_1}(x) \end{aligned}$$

with, $\sigma_0(\xi) = \cos(2\pi t|\xi|)$ and $\sigma_1(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}$.

We recall that H_{σ_i} $i = 0, 1$, are examples of Fourier multipliers which are defined by

$$H_\sigma f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi} \sigma(\xi) \hat{f}(\xi)\, d\xi \tag{5.2}$$

where σ is called the symbol.

The boundedness of H_{σ_i}, $i = 0, 1$ on modulation spaces was proved in [4, 3] and in [6]. Moreover, some related local-in-time well posedness results for certain nonlinear PDEs were also obtained in [3, 6] for initial data in modulation spaces.

Proposition 5.1. *Let $s \in \mathbb{R}$, and $1 \le p, q \le \infty$. Then, the solution $u(t, x)$ of (5.1) with initial data $(u_0, u_1) \in \mathcal{M}^{p,q}_{0,s} \times \mathcal{M}^{p,q}_{0,s-1}$ satisfies*

$$\|u(t, \cdot)\|_{\mathcal{M}^{p,q}_{0,s}} \le C_0(1 + t)^{d+1}\|u_0\|_{\mathcal{M}^{p,q}_{0,s}} + C_1 t(1 + t)^{d+1}\|u_1\|_{\mathcal{M}^{p,q}_{0,s-1}} \tag{5.3}$$

where C_0 and C_1 are only functions of the dimension d.

Proof. It was proved in [4] that $\sigma_0(\xi) \in W(\mathcal{F}L^1, L^\infty)$ and in [6] that $\sigma_1(\xi) \in W(\mathcal{F}L^1, L^\infty_1)$. In addition, it was shown in [6] that the solution satisfies

$$\begin{aligned}
\|u(t, \cdot)\|_{\mathcal{M}^{p,q}_{0,s}} &\le \|H_{\sigma_0} u_0\|_{\mathcal{M}^{p,q}_{0,s}} + \|H_{\sigma_1} u_1\|_{\mathcal{M}^{p,q}_{0,s}} \\
&\le \|H_{\sigma_0} u_0\|_{\mathcal{M}^{p,q}_{0,s}} + \|H_{\sigma_1} u_1\|_{\mathcal{M}^{p,q}_{0,s-1}} \\
&\le \|\sigma_0\|_{W(\mathcal{F}L^1, L^\infty)}\|u_0\|_{\mathcal{M}^{p,q}_{0,s}} + \|\sigma_1\|_{W(\mathcal{F}L^1, L^\infty_1)}\|u_1\|_{\mathcal{M}^{p,q}_{0,s-1}} \\
&\le C_0(t)\|u_0\|_{\mathcal{M}^{p,q}_{0,s}} + C_1(t)\|u_1\|_{\mathcal{M}^{p,q}_{0,s-1}}.
\end{aligned}$$

We can now use the results proved in Section 3 to estimate $C_0(t)$ and $C_1(t)$. More specifically, setting $\widetilde{\sigma_0}(\xi) = \cos|\xi|$, for $t > 0$, we can write $\sigma_0(\xi) = (\widetilde{\sigma_0})_{2\pi t}$. Using (3.5) with $\mu_1(\infty, 1) = 1$, $\mu_2(\infty, 1) = 0$, we have, for every $R > 0$,

$$\|(\widetilde{\sigma_0})_{2\pi t}\|_{W(\mathcal{F}L^1, L^\infty_1)} \le \begin{cases} C_{0,R}\|\widetilde{\sigma_1}\|_{W(\mathcal{F}L^1, L^\infty)}, & t \le R \\ C'_{0,R} t^{d+1}\|\widetilde{\sigma_0}\|_{W(\mathcal{F}L^1, L^\infty)}, & t \ge R. \end{cases}$$

Hence

$$C_0(t) \le \begin{cases} C_{0,R}, & 0 \le t \le R \\ C'_{0,R} t^{d+1}, & t \ge R. \end{cases}$$

Setting $\widetilde{\sigma_1}(\xi) = \frac{\sin|\xi|}{|\xi|}$, for $t > 0$, we can write $\sigma_1(\xi) = t(\widetilde{\sigma_1})_{2\pi t}$ and, for every $R > 0$,

$$\|(\widetilde{\sigma_1})_{2\pi t}\|_{W(\mathcal{F}L^1, L^\infty_1)} \le \begin{cases} C_{1,R}\|\widetilde{\sigma_1}\|_{W(\mathcal{F}L^1, L^\infty_1)}, & t \le R \\ C'_{1,R} t^{d+1}\|\widetilde{\sigma_1}\|_{W(\mathcal{F}L^1, L^\infty_1)}, & t \ge R. \end{cases}$$

Hence

$$C_1(t) \le \begin{cases} C_{1,R} t, & 0 \le t \le R \\ C'_{1,R} t^{d+2}, & t \ge R, \end{cases}$$

and the estimate (5.3) becomes

$$\|u(t, \cdot)\|_{\mathcal{M}^{p,q}_{0,s}} \le C_0(1 + t)^{d+1}\|u_0\|_{\mathcal{M}^{p,q}_{0,s}} + C_1 t(1 + t)^{d+1}\|u_1\|_{\mathcal{M}^{p,q}_{0,s-1}}, \quad t > 0.$$

□

5.1.2. *Vibrating plate equation.* Consider now the following Cauchy problem for the vibrating plate equation

$$\begin{cases} \partial_t^2 u + \Delta_x^2 u = 0 \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \end{cases} \tag{5.4}$$

with $t \geq 0$, $x \in \mathbb{R}^d$, $d \geq 1$. The formal solution $u(t,x)$ is given by

$$u(t,x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi} \cos(4\pi^2 t|\xi|^2)\widehat{u_0}(\xi)\,d\xi + \int_{\mathbb{R}^d} e^{2\pi i x\xi} \frac{\sin(4\pi^2 t|\xi|^2)}{4\pi^2|\xi|^2}\widehat{u_1}(\xi)\,d\xi,$$

and satisfies the following estimate.

Proposition 5.2. *Let $s \in \mathbb{R}$, and $1 \leq p, q \leq \infty$. Then, the solution $u(t,x)$ of* (5.4) *with initial data $(u_0, u_1) \in \mathcal{M}^{p,q}_{0,s} \times \mathcal{M}^{p,q}_{0,s-2}$ satisfies*

$$\|u(t,\cdot)\|_{\mathcal{M}^{p,q}_{0,s}} \leq C_0(1+t)^{\frac{d}{2}}\|u_0\|_{\mathcal{M}^{p,q}_{0,s}} + C_1 t(1+t)^{\frac{d}{2}+1}\|u_1\|_{\mathcal{M}^{p,q}_{0,s-2}} \tag{5.5}$$

where C_0 and C_1 are only functions of the dimension d.

Proof. Here the solution is the sum of two Fourier multipliers $u = H_0u_0 + H_1u_1$ having symbols $\sigma_0(\xi) = \cos(4\pi^2 t|\xi|^2) \in W(\mathcal{F}L^1, L^\infty)$ (see [4]) and $\sigma_1(\xi) = \frac{\sin(4\pi^2 t|\xi|^2)}{4\pi^2|\xi|^2} \in W(\mathcal{F}L^1, L^\infty_2)$ (see [5]).

Since $\sigma_0(\xi) = \cos(|\xi|^2)_{2\pi\sqrt{t}}$ and $\sigma_1(\xi) = t\left(\frac{\sin(|\xi|^2)}{|\xi|^2}\right)_{2\pi\sqrt{t}}$, using the same arguments as for the wave equation we obtain:

$$\|u(t,\cdot)\|_{\mathcal{M}^{p,q}_{0,s}} \leq C_0(1+t)^{\frac{d}{2}}\|u_0\|_{\mathcal{M}^{p,q}_{0,s}} + C_1 t(1+t)^{\frac{d}{2}+1}\|u_1\|_{\mathcal{M}^{p,q}_{0,s-2}}, \quad t > 0.$$

□

5.2. **Embedding of Besov spaces into modulation spaces.** We generalize some results of [10]. But first, we recall the inclusion relations between Besov spaces and modulation spaces (see [14, 1]). Consider the following indices, where μ_i, $i = 1, 2$ were defined in Section 2:

$$\nu_1(p,q) = \mu_1(p,q) + \frac{1}{p}, \qquad \nu_2(p,q) = \mu_2(p,q) + \frac{1}{p}.$$

The following result was proved in [15, Theorem 3.1] and in [1, Theorem 1.1]

Theorem 5.3. *Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.*
(i) If $s \geq d\nu_1(p,q)$ then $B^{p,q}_s(\mathbb{R}^d) \hookrightarrow \mathcal{M}^{p,q}(\mathbb{R}^d)$.
(ii) If $s \leq d\nu_2(p,q)$ then $\mathcal{M}^{p,q}(\mathbb{R}^d) \hookrightarrow B^{p,q}_s(\mathbb{R}^d)$.

The next results improve those in [10, Theorem 3.1].

Theorem 5.4. *Let $1 \leq p \leq 2$.*
(i) If $s \geq d(1/p - 1/p')$ and $1 \leq q \leq p$ then $B^{p,q}_s \hookrightarrow \mathcal{M}^p$.
(ii) If $s > d(1/p - 1/p')$ and $1 \leq q \leq \infty$ then $B^{p,q}_s \hookrightarrow \mathcal{M}^p$.

Proof. (i) For $s \geq d(1/p - 1/p') \geq \nu_1(p,p) = 0$, Theorem 5.3 says that $B_s^{p,p} \hookrightarrow \mathcal{M}^{p,p}$. However, the inclusion relations for Besov spaces give $B_s^{p,q} \hookrightarrow B_s^{p,p}$, for $q \leq p$. Hence the result follows.

(ii) If $s > d(1/p - 1/p') \geq 0$, and $q \leq p$, then this is exactly (i) above. If $p \leq q$, then $B_s^{p,q} \hookrightarrow B^{p,p} \hookrightarrow \mathcal{M}^p$. □

The next results improve those in [10, Theorem 3.2].

Theorem 5.5. *(i) Let* $1 \leq p \leq 2$, $s > 0$. *Then* $B_s^{p,q} \hookrightarrow \mathcal{M}^{p,p'}$, *for all* $1 \leq q \leq \infty$.
(ii) If $2 \leq p \leq \infty$, $s > d(1/p' - 1/p)$, *then* $B_s^{p,q} \hookrightarrow \mathcal{M}^{p,p'}$, *for all* $1 \leq q \leq \infty$.

Proof. (i) For $1 \leq p \leq 2$, $\nu_1(p,p') = 0$ and using Theorem 5.3 we obtain $B^{p,p'} \hookrightarrow \mathcal{M}^{p,p'}$. Since $B_s^{p,q} \hookrightarrow B^{p,p'}$, for all $1 \leq q \leq \infty$, $s > 0$, the result follows.
(ii) If $2 \leq p \leq \infty$,

$$\nu_1(p,p') = \frac{1}{p'} - \frac{1}{p} \leq \frac{1}{p'}.$$

Hence, if $s \geq d(1/p' - 1/p)$, Theorem 5.3 gives $B_s^{p,p'} \hookrightarrow \mathcal{M}^{p,p'}$. If $s > d(1/p' - 1/p)$, the inclusion relations for Besov spaces give $B_s^{p,q} \hookrightarrow B^{p,p'}_{d(1/p'-1/p)}$. This is easy to see if $q \leq p'$. On the other hand if $q > p'$ it follows by an application of Hölder's inequality for ℓ^p spaces. In any case, this concludes the proof. □

6. Acknowledgment

The authors would like to thank Fabio Nicola for helpful discussions. K. A. Okoudjou would also like to acknowledge the partial support of the Alexander von Humboldt foundation.

References

[1] W. Baoxiang and H. Chunyan. Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations. *J. Differential Equations*, 239(1):213–250, 2007.
[2] Á. Bényi, K. Gröchenig, C. Heil and K. Okoudjou. Modulation spaces and a class of bounded multilinear pseudodifferential operators. *J. Operator Theory*, 54:387–399, 2005.
[3] Á. Bényi and K. A. Okoudjou. Local well-posedness of nonlinear dispersive equations on modulation spaces. *Bull. Lond. Math. Soc.*, 41:549–558, 2009.
[4] Á. Bényi, K. Gröchenig, K. A. Okoudjou, and L. G. Rogers. Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.*, 246(2):366-384, 2007.
[5] E. Cordero and D. Zucco. The Cauchy Problem for the Vibrating Plate Equation in Modulation Spaces. *Preprint*, 2010. Available at http://arxiv.org/abs/1004.3686.
[6] E. Cordero and F. Nicola. Remarks on Fourier multipliers and applications to the Wave equation, *J. Math. Anal. Appl.*, 353(2):583–591, 2009.
[7] H. G. Feichtinger. Modulation spaces on locally Abelian groups. Technical Report, University of Vienna, 1983, Updated version appeared in Proceedings of "International Conference on Wavelets and Applications" 2002, pp. 99–140, Chennai, India, 2003.
[8] H. G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. In *Proc. Conf. Constructive Function Theory*, *Rocky Mountain J. Math.*, 19:113–126, 1989.
[9] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Birkhäuser, Boston, 2001.

[10] K. A. Okoudjou. Embeddings of some classical Banach spaces into modulation spaces. *Proc. Amer. Math. Soc.*, 132:1639–1647, 2004.
[11] K. A. Okoudjou. A Beurling-Helson type theorem for modulation spaces. *J. Funct. Spaces Appl.*, 7(1):33–41, 2009.
[12] T. Runst and W. Sickel. *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations,* Walter de Gruyter, Berlin, 1996.
[13] M. A.Shubin. *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin, 1987.
[14] M. Sugimoto and N. Tomita. The dilation property of modulation spaces and their inclusion relation with Besov spaces. *J. Funct. Anal.*, 248:79–106, 2007.
[15] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I. *J. Funct. Anal.*, 207:399–429, 2004.
[16] J. Toft. Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II. *Ann. Global Anal. Geom.*, 26(1):73–106, 2004.
[17] H. Triebel. Modulation spaces on the euclidean $n-$space. *Z. Anal. Anwendungen*, 2:443–457, 1983.

Department of Mathematics, University of Torino, Via Carlo Alberto 10, 10123 Torino, Italy

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

E-mail address: elena.cordero@unito.it
E-mail address: kasso@math.umd.edu